Supplemental Material for "Sticky Discount Rates"

Supplement A Derivations and Proofs

Supplement A.1 Proof of Proposition 1

We approximate around the steady state where all the exogenous variables are constant over time. We denote the steady state values of all the variables as those without time subscript. The existence of such a steady state requires

$$q = \frac{\omega - \xi}{r} = 1, \tag{S1}$$

which we assume to hold. In general equilibrium, ω endogenously adjusts to ensure the above property, but here, we simply assume that $\omega = r + \xi$. The investment rate and the discount rate are

$$\iota = \xi, \tag{S2}$$

$$\delta = r + \pi. \tag{S3}$$

We denote log deviations from the steady state value using hats (e.g., $\hat{x}_t = \log(x_t/x)$).

We start by linearizing the investment policies (8). The first-order approximation of (8) is

$$\iota_{it} - \xi = \frac{1}{\phi}\left[\hat{q}_t - \frac{1}{r}(\delta_{it} - coc_t)\right], \tag{S4}$$

where

$$\hat{q}_t = \int_t^\infty e^{-r(s-t)}(\omega_s - r_s - \xi)ds \tag{S5}$$

and

$$coc_t = r\int_t^\infty e^{-r(s-t)} i_s ds = r\int_t^\infty e^{-r(s-t)}(r_s + \pi_s)ds. \tag{S6}$$

The key observation is that future investment policies $\{\iota_{is}\}_{s>t}$ have no first-order effects around the efficient steady state thanks to the envelope theorem. That is,

$$\frac{\partial}{\partial \iota_{is}} q_t^\delta(\delta_{it}; \{\iota_{is}\}_{s>t}) = 0 \tag{S7}$$

for all $s > t$ when evaluated around the efficient steady state. We can therefore write the investment policy function of a firm as a function of macro aggregates and its current discount rate:

$$\iota_t(\delta) = \xi + \frac{1}{\phi}\left[\hat{q}_t - \frac{1}{r}(\delta_{it} - coc_t)\right]. \tag{S8}$$

The aggregate investment expression follows by aggregating the above expression across $i \in [0, 1]$.

Supplement A.2 Proof of Proposition 2

The unit value of capital of firm i at time t is

$$\int_t^\infty e^{-\int_0^s [i_{t+u} - \pi_{t+u} - (\iota_{it+u} - \xi)]du} \left[\omega_s - \iota_{is} + \varphi(\iota_{is})\right] ds. \tag{S9}$$

The second-order approximation of the above expression around the efficient steady state is

$$\int_t^\infty e^{-r(s-t)} \left[-\frac{\phi}{2}(\iota_{is} - \iota)^2 + \hat{q}_{is}(\iota_{is} - \iota)\right] ds + t.i.p., \tag{S10}$$

where $t.i.p.$ refers to terms independent of (investment) policies and $\hat{q}_{it}$ is defined in (S5).

We define the linear-quadratic approximation of the firm's discount rate setting problem as follows:

$$\max_{\delta^*} \int_t^\infty e^{-(r+\theta^\delta_{f(i)})(s-t)} \left[-\frac{\phi}{2}(\iota_s(\delta^*) - \iota)^2 + \hat{q}_s(\iota_s(\delta^*) - \iota)\right] ds, \tag{S11}$$

where $\iota_t(\delta^*)$ is given by (S8). This problem reflects that the firm can adjust the discount rate with Poisson arrival rate $\theta^\delta_{f(i)}$. Until the firm is able to adjust, it must follow the investment strategy given by (S8). Since the objective function does not involve first-order terms, the above linear-quadratic approximation provides a valid approximation to the original problem (Benigno and Woodford 2012).

The first-order condition of the above problem is

$$\int_t^\infty e^{-(r+\theta^\delta_f)(s-t)} \left[-\phi(\iota_s(\delta^*_{ft}) - \iota) + \hat{q}_s\right] \iota'_s(\delta^*_{ft}) ds = 0 \tag{S12}$$

for a firm i in group f. Using (S8), we can rewrite the above expression as

$$\int_0^\infty e^{-(r+\theta_f)s}\left(\delta^*_{ft} - coc_s\right) ds = 0, \tag{S13}$$

which implies (15). The average discount rate of group f evolves as

$$\partial_t \delta_{ft} = \theta_{ft}(\delta^*_{ft} - \delta_{ft}), \tag{S14}$$

which follows from the fact that a randomly selected subset of firms adjusts their discount rates to δ^*_{ft}. This is equivalent to (16). Equation (17) then follows from aggregating across all firm groups.

Supplement A.3 Proof of Proposition 3

To a first order, the change in the path of inflation is

$$d\pi_t = e^{-\beta_\pi t} d\pi_0, \tag{S15}$$

where $d\pi_0 > 0$. Using (9), the change in the cost of capital is

$$dcoc_t = \frac{r}{r+\beta_\pi} e^{-\beta_\pi t} d\pi_0. \tag{S16}$$

Substituting the above expression into (15), we have

$$d\delta^*_{ft} = \frac{r}{r+\beta_\pi} \frac{r+\theta^\delta_f}{r+\theta^\delta_f+\beta_\pi} e^{-\beta_\pi t} d\pi_0. \tag{S17}$$

Accordingly, using (16), we can compute the group-level discount rate as

$$d\delta_{ft} = \frac{r}{r+\beta_\pi} \frac{r+\theta^\delta_f}{r+\theta^\delta_f+\beta_\pi} \theta^\delta_f \frac{1}{\theta^\delta_f - \beta_\pi} \left[e^{-\beta_\pi t} - e^{-\theta^\delta_f t} \right] d\pi_0. \tag{S18}$$

The discount rate wedge is

$$d\delta_{ft} - dcoc_t = \frac{r}{r+\beta_\pi} e^{-\beta_\pi t} \left(\frac{r+\theta^\delta_f}{r+\theta^\delta_f+\beta_\pi} \theta^\delta_f \frac{1}{\theta^\delta_f - \beta_\pi} \left[1 - e^{(\beta_\pi - \theta^\delta_f)t} \right] - 1 \right) d\pi_0. \tag{S19}$$

Since $\hat{q}_t = 0$, equation (10) implies

$$d\iota_t = -\frac{1}{r\phi}(d\delta_{ft} - dcoc_t), \tag{S20}$$

so that the response of the investment rate is positive if the discount rate wedge is negative. The discount rate wedge is negative if

$$t < \frac{1}{\theta_f^\delta - \beta_\pi} \log\left[\frac{(r+\theta_f^\delta)\theta_f^\delta}{(r+\beta_\pi)\beta_\pi}\right]. \tag{S21}$$

The cumulative impulse response of the investment rate is

$$\begin{aligned}
\int_0^\infty d\iota_t dt &= -\int_0^\infty \frac{1}{r\phi}\left[d\delta_{ft} - dcoc_t\right] dt & \text{(S22)}\\
&= -\frac{1}{\phi r}\frac{r}{r+\beta_\pi}\left(\frac{r+\theta_f^\delta}{r+\theta_f^\delta+\beta_\pi}\frac{1}{\beta_\pi} - \frac{1}{\beta_\pi}\right) d\pi_0 & \text{(S23)}\\
&> 0. & \text{(S24)}
\end{aligned}$$

With flexible discount rates ($\theta_f^\delta = \infty$), the discount rate wedge is zero for all t. Therefore, the investment rate response is zero for all t as well.

Supplement A.4 Proof of Proposition 4

The proof is nearly identical to the proof of Proposition 3 above. To a first order, the change in the path of the real interest rate is

$$dr_t = e^{-\beta_r t} dr_0. \tag{S25}$$

Using (9), the change in cost of capital is

$$dcoc_t = \frac{r}{r+\beta_r} e^{-\beta_r t} dr_0. \tag{S26}$$

Substituting the above expression into (15), we have

$$d\delta_{ft}^* = \frac{r}{r+\beta_r}\frac{r+\theta_f^\delta}{r+\theta_f^\delta+\beta_r} e^{-\beta_r t} dr_0. \tag{S27}$$

Accordingly, using (16), we can compute the group-level discount rate as

$$d\delta_{ft} = \frac{r}{r+\beta_r}\frac{r+\theta_f^\delta}{r+\theta_f^\delta+\beta_r}\theta_f^\delta\frac{1}{\theta_f^\delta-\beta_r}\left[e^{-\beta_r t}-e^{-\theta_f^\delta t}\right]dr_0. \tag{S28}$$

The discount rate wedge is

$$d\delta_{ft} - dcoc_t = \frac{r}{r+\beta_r}e^{-\beta_r t}\left(\frac{r+\theta_f^\delta}{r+\theta_f^\delta+\beta_r}\theta_f^\delta\frac{1}{\theta_f^\delta-\beta_r}\left[1-e^{(\beta_r-\theta_f^\delta)t}\right]-1\right)dr_0. \tag{S29}$$

The difference in the response of the investment rate under flexible discount rates ($d\iota_t^{flex}$) and sticky discount rates ($d\iota_t$) is proportional to the discount rate wedge and given by

$$d\iota_t - d\iota_t^{flex} = -\frac{1}{\phi r}(d\delta_{ft} - dcoc_t). \tag{S30}$$

One can compute that $d\iota_t - d\iota_t^{flex} > 0$ if and only if

$$t < \frac{1}{\theta_f^\delta-\beta_r}\log\left[\frac{(r+\theta_f^\delta)\theta_f^\delta}{(r+\beta_r)\beta_r}\right]. \tag{S31}$$

Furthermore, the difference in the cumulative impulse response of the investment rate is

$$\begin{aligned}\int_0^\infty (d\iota_t - d\iota_t^{flex})dt &= -\int_0^\infty \frac{1}{r\phi}\left[d\delta_{ft} - dcoc_t\right]dt &\text{(S32)}\\ &= -\frac{1}{\phi r}\frac{r}{r+\beta_r}\left(\frac{r+\theta_f^\delta}{r+\theta_f^\delta+\beta_r}\frac{1}{\beta_r}-\frac{1}{\beta_r}\right)dr_0 &\text{(S33)}\\ &> 0. &\text{(S34)}\end{aligned}$$

Supplement A.5 Derivation of Firm's Loss Function

Substituting (10) into (S10), we obtain firm profits as

$$\begin{aligned}&\mathbb{E}_t\int_t^\infty e^{-r(s-t)}\left[-\frac{\phi}{2}(\iota_{is}-\iota)^2-\hat{q}_{is}(\iota_{is}-\iota)\right]ds\\ =&-\mathbb{E}_t\frac{1}{2}\frac{1}{\phi}\int_t^\infty e^{-r(s-t)}\left[-\hat{q}_s^2+\frac{1}{r^2}(\delta_{is}-coc_s)^2\right]ds.\end{aligned}$$

Since $\hat{q}_t$ is independent of the firm's policy, the change in firm profits from sticky discount rates is

$$-\mathbb{E}_t \frac{1}{2}\frac{1}{\phi r^2}\int_t^\infty e^{-r(s-t)}(\delta_{is} - coc_s)^2 ds.$$

Supplement A.6 List of Equilibrium Conditions

Given a sequence of shocks $\{\rho_t, \hat{A}_t\}$ and monetary policies $\{r_t, \pi_\infty\}$, the equilibrium consists of $\{coc_t, \omega_t, \delta^*_{ft}, \delta_{ft}, \delta_t, \iota_t, q_t, \hat{C}_t, \pi_t, \hat{L}_t, \hat{K}_t\}$ that solve

$$\partial_t coc_t = -r(r_t + \pi_t) + r coc_t \tag{S35}$$

$$(r + \theta^\delta_f)\delta^*_{ft} = (r + \theta^\delta_f) coc_t + \partial_t \delta^*_{ft} \tag{S36}$$

$$\partial_t \delta_t = \theta^\delta_f(\delta^*_{ft} - \delta_{ft}) \tag{S37}$$

$$\delta_t = \mathbb{E}_f[\delta_{ft}] \tag{S38}$$

$$r\hat{q}_t = \omega\hat{\omega}_t - (r_t - r) + \partial_t \hat{q}_t \tag{S39}$$

$$\iota_t = \frac{1}{\phi}\left[\hat{q}_t - \frac{1}{r}(\delta_t - coc_t)\right] \tag{S40}$$

$$\hat{\omega}_t = \hat{A}_t - (1-\alpha)(\hat{K}_t - \hat{L}_t) \tag{S41}$$

$$\partial_t \hat{C}_t = \frac{1}{\sigma}(r_t - \rho_t) \tag{S42}$$

$$\partial_t \pi_t = -\theta^p(r + \theta^p)\left[\sigma\hat{C}_t + \nu\hat{L}_t - \hat{A}_t - \alpha(\hat{K}_t - \hat{L}_t)\right] + r\pi_t \tag{S43}$$

$$\frac{C}{Y}\hat{C}_t + \frac{K}{Y}(\iota_t - \xi + \iota\hat{K}_t) = \hat{A}_t + \alpha\hat{K}_t + (1-\alpha)\hat{L}_t \tag{S44}$$

$$\partial_t \hat{K}_t = \iota_t - \xi. \tag{S45}$$

Supplement A.7 Derivation of Central Bank Objective

We will often invoke the following second-order approximation:

$$\frac{x_t - x}{x} \approx \hat{x}_t + \frac{1}{2}\hat{x}_t^2. \tag{S46}$$

We define welfare in the economy as

$$W = \int_0^\infty e^{-\int_0^t \rho_s ds}\left[u(C_t) - v(L_t)\right] dt. \tag{S47}$$

The second-order approximation of (S47) around the steady state is

$$
\begin{aligned}
W \approx \int_0^\infty e^{-\rho t} \Bigg[& \bar{u}'(C)C \left(\hat{C}_t + \frac{1}{2}(1-\sigma)\hat{C}_t^2 \right) \\
& - v'(L)L \left(\hat{L}_t + \frac{1+\nu}{2}\hat{L}_t^2 \right) - \bar{u}'(C)C \int_0^t \rho_s ds \hat{C}_t + v'(L)L \int_0^t \hat{\rho}_s ds \hat{L}_t \Bigg] + t.i.p., \quad \text{(S48)}
\end{aligned}
$$

where $t.i.p.$ denotes a set of terms independent of policies. The resource constraint is

$$
\int \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} di \left[C_t + \iota_t K_t + \int \varphi \left(\frac{\iota_t(i)}{\iota_t} \iota_t \right) di K_t \right] = A_t F\left(K_t, L_t\right), \tag{S49}
$$

where i indexes a price-setting firm. Let $\hat{p}_t(i) = \log P_t(i) - \log P_t$ and $\hat{w}_t(\ell) = \log W_t(\ell) - \log W_t$.

The misallocation resulting from price dispersion can be expressed as:

$$
\int \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} di \approx 1 - \varepsilon \int \hat{p}_t(i) di + \frac{\varepsilon^2}{2} \int \hat{p}_t(i)^2 di. \tag{S50}
$$

Since $\int \left(\frac{P_t(i)}{P_t} \right)^{1-\varepsilon} di = 1$ by the definition of the price index, we also have

$$
1 \approx 1 + (1-\varepsilon) \int \hat{p}_t(i) di + \frac{(1-\varepsilon)^2}{2} \int \hat{p}_t(i)^2 di. \tag{S51}
$$

Combining the previous two expressions, price dispersion is

$$
\begin{aligned}
\int \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} di - 1 &\approx \frac{\varepsilon}{2} \int \hat{p}_t(i) di & \text{(S52)} \\
&= \frac{\varepsilon}{2} \mathrm{var}(\hat{p}_t(i)). & \text{(S53)}
\end{aligned}
$$

In a similar vein, we can define $\hat{\iota}^d(i) = \log(\iota_t(i)/\iota_t)$ and express the misallocation from investment dispersion as

$$
\begin{aligned}
\int \varphi \left(\frac{\iota_t(i)}{\iota_t} \iota_t \right) di &\approx \frac{1}{2} \varphi''\left(\iota\right) \iota^2 \int \hat{\iota}^d(i)^2 di + \frac{1}{2} \varphi''\left(\iota\right) \iota^2 \hat{\iota}_t^2 \\
&= \frac{1}{2} \varphi''\left(\iota\right) \iota^2 \mathrm{var}(\hat{\iota}^d(i)) + \frac{1}{2} \varphi''\left(\iota\right) \iota^2 \hat{\iota}_t^2.
\end{aligned} \tag{S54}
$$

Therefore, the second-order approximation of (S49) is

$$C\left(\hat{C}_t+\frac{1}{2}\hat{C}_t^2\right)+I\left(\hat{I}_t+\frac{1}{2}\hat{I}_t^2\right)+\frac{1}{2}\phi K\iota^2\mathrm{var}(\hat{\iota}^d(i))+\frac{1}{2}\phi K\iota^2\hat{\iota}_t^2+G_t+\frac{1}{2}G_t^2 = Y\left[-\frac{\varepsilon}{2}\mathrm{var}(\hat{p}_t(i))+\hat{Y}_t+\frac{1}{2}\hat{Y}_t^2\right]. \tag{S55}$$

The second-order approximation of the Cobb-Douglas production function gives

$$\hat{Y}_t+\frac{1}{2}\hat{Y}_t^2=\hat{A}_t+\frac{1}{2}(\hat{A}_t)^2+\alpha(\hat{K}_t+\frac{1}{2}\hat{K}_t^2)+(1-\alpha)(\hat{L}_t+\frac{1}{2}(\hat{L}_t)^2)-\frac{1}{2}\alpha(1-\alpha)(\hat{L}_t-\hat{K}_t)^2 + (1-\alpha)\hat{A}_t\hat{L}_t+\alpha\hat{A}_t\hat{K}_t. \tag{S56}$$

The second-order approximation of the capital accumulation equation, $\partial_t K_t=(\iota_t-\xi)K_t$, is

$$K\left(\partial_t\hat{K}_t+\frac{1}{2}\partial_t\hat{K}_t^2\right)=I\left(\hat{I}_t+\frac{1}{2}\hat{I}_t^2\right)-\xi K\left(\hat{K}_t+\frac{1}{2}\hat{K}_t^2\right), \tag{S57}$$

where $I_t=\iota_t K_t$ denotes aggregate investment.

Using (S55), one can rewrite $\bar{u}'(C)C\hat{C}_t$ as

$$\bar{u}'(C)C\hat{C}_t=\bar{u}'(C)\Bigg\{Y\Bigg[-\frac{\varepsilon}{2}\mathrm{var}(\hat{p}_t(i))+\alpha(\hat{K}_t+\frac{1}{2}\hat{K}_t^2)+(1-\alpha)(\hat{L}_t+\frac{1}{2}\hat{L}_t^2) -\frac{1}{2}\alpha(1-\alpha)(\hat{L}_t-\hat{K}_t)^2\Bigg]-\frac{1}{2}C\hat{C}_t^2-I\left(\hat{I}_t+\frac{1}{2}\hat{I}_t^2\right)-\frac{1}{2}\varphi''(\iota)\,\iota^2K\mathrm{var}(\hat{\iota}^d(i)) -\frac{1}{2}\varphi''(\iota)\,\iota^2K\hat{\iota}_t^2+Y\left[(1-\alpha)\hat{A}_t\hat{L}_t+\alpha\hat{A}_t\hat{K}_t\right]\Bigg\}+t.i.p. \tag{S58}$$

Substituting (S58) and (S57) into (S48),

$$W\approx\bar{u}'(C)\int_0^\infty e^{-\rho t}\Bigg[-Y\frac{\varepsilon}{2}\mathrm{var}(\hat{p}_t(i))-\frac{1}{2}\alpha(1-\alpha)Y(\hat{L}_t-\hat{K}_t)^2 -\frac{1}{2}\phi\xi^2K\mathrm{var}(\hat{\iota}^d(i))-\frac{1}{2}\phi\xi^2K(\hat{I}_t-\hat{K}_t)^2-\frac{1}{2}\sigma C\hat{C}_t^2-(1-\alpha)Y\frac{\nu}{2}\hat{L}_t^2 . -\bar{u}'(C)C\int_0^t\rho_s ds\hat{C}_t+(1-\alpha)Y\int_0^t\hat{\rho}_s ds\hat{L}_t+(1-\alpha)Y\hat{A}_t\hat{L}_t+\alpha Y\hat{A}_t\hat{K}_t\Bigg]dt+t.i.p., \tag{S59}$$

In deriving the above, we have used the fact that

$$(1-\alpha)Y\hat{L} - v'(L)L\hat{L}_t = 0 \tag{S60}$$

and

$$\begin{aligned}(\alpha Y - \xi K)K\int e^{-\rho t}x_t dt - K\int e^{-\rho t}\partial_t x_t dt &= \rho K\int e^{-\rho t}x_t dt - K\int e^{-\rho t}\partial_t x_t dt && \text{(S61)}\\ &= 0, && \text{(S62)}\end{aligned}$$

where $x \equiv \hat{K}_t + \frac{1}{2}\hat{K}_t^2$ and the last equality follows from integration by parts.

$$\bar{u}'(C)I\hat{I}_t = \bar{u}'(C)\left[K\left(\hat{K}_{t+1} + \frac{1}{2}\hat{K}_{t+1}^2\right) - (1-\xi)K\left(\hat{K}_t + \frac{1}{2}\hat{K}_t^2\right) - I\frac{1}{2}\hat{I}_t^2\right]. \tag{S63}$$

Price dispersion evolves according to (see, e.g., Woodford (2003) for the derivation),

$$\partial_t \text{var}(\hat{p}_t(i)) = -\theta^p \text{var}(\hat{p}_t(i)) + \frac{1}{\theta^p}\hat{\pi}_t^2. \tag{S64}$$

Using integration by parts,

$$\begin{aligned}\int_0^\infty e^{-\rho t}\text{var}(\hat{p}_t(i))dt &= -\left[\frac{1}{\rho}e^{-\rho t}\text{var}(\hat{p}_t(i))\right]_0^\infty + \int_0^\infty \frac{1}{\rho}e^{-\rho t}\frac{d}{dt}\text{var}(\hat{p}_t(i))dt && \text{(S65)}\\ &= \int_0^\infty \frac{1}{\rho}e^{-\rho t}\left(-\theta^p\text{var}(\hat{p}_t(i)) + \frac{1}{\theta^p}\hat{\pi}_t^2\right)dt && \text{(S66)}\\ &= -\frac{\theta}{\rho}\int_0^\infty e^{-\rho t}\text{var}(\hat{p}_t(i))dt + \frac{1}{\rho\theta^p}\int e^{-\rho t}\hat{\pi}_t^2 dt, && \text{(S67)}\end{aligned}$$

which we can solve as

$$\int_0^\infty e^{-\rho t}\text{var}(\hat{p}_t(i))dt = \frac{1}{(\rho+\theta^p)\theta^p}\int e^{-\rho t}\hat{\pi}_t^2 dt. \tag{S68}$$

Next, we seek to express the investment misallocation term, $\text{var}(\hat{\iota}_t^d(i))$. Recall that the investment rate is

$$\hat{\iota}_t = \frac{1}{\phi\xi}\left[q_t - \frac{1}{r}(\delta_t - coc_t)\right]. \tag{S69}$$

Investment misallocation can be written as the dispersion in discount rates,

$$\text{var}(\hat{\iota}_t^d(i)) = \frac{1}{(r\phi\xi)^2}\text{var}(\hat{\delta}_t(i)). \tag{S70}$$

The evolution of the aggregate discount rate is dictated by

$$\partial_t \delta_{ft} = \theta_f^{\delta}(\delta_{ft}^* - \delta_{ft}), \tag{S71}$$

where δ_t^* denotes the discount rate of firms with an adjustment opportunity in period t.

We can rewrite $\text{var}(\delta^d(i))$ as

$$\text{var}(\delta_t(i)) = \sum_f \ell_f \text{var}_{i|f}(\delta_t(i))di + \sum_f \ell_f(\delta_{ft} - \delta_t)^2, \tag{S72}$$

where $\text{var}_{i|f}(x) \equiv \frac{1}{\ell f}\int_{i\in f}(x_i - x_f)^2$ and $x_f = \frac{1}{\ell f}\int_{i\in f} x_i di$ are variance and expectation operators conditional on group f for an arbitrary variable x. The evolution of $\text{var}(\delta_t(i))$ is given by

$$\partial_t \text{var}_{i|f}(\delta_t(i)) = -\theta_f^{\delta}\text{var}_{i|f}(\delta_t(i)) + \frac{1}{\theta_f^{\delta}}\left(\partial_t \delta_{ft}\right)^2. \tag{S73}$$

Using the above relationship, we can write

$$\int_0^\infty e^{-\rho t}\text{var}_{i|f}(\delta_t(i))dt = \frac{1}{(\rho + \theta_f^{\delta})\theta_f^{\delta}}\int_0^\infty e^{-\rho t}(\partial_t \delta_{ft})^2 dt. \tag{S74}$$

Consequently, expression (S70) becomes

$$\int_0^\infty e^{-\rho t}\text{var}(\hat{\iota}_t^d(i))dt = \int_0^\infty \frac{1}{(r\phi\xi)^2}e^{-\rho t}\left[\sum_f \ell_f \frac{1}{(\rho + \theta_f^{\delta})\theta_f^{\delta}}(\partial_t \delta_{ft})^2 + \sum_f \ell_f(\delta_{ft} - \delta_t)^2\right]dt. \tag{S75}$$

Substituting (S68) and (S75) into (S59), the quadratic loss function is

$$W \approx -\frac{u'(C)Y}{2}\int_0^\infty e^{-\rho t}\mathbb{L}_t dt, \tag{S76}$$

where

$$\mathbb{L}_t \equiv \Bigg[\varpi_{KL}(\hat{L}_t - \hat{K}_t)^2 + \varpi_{IK}\iota_t^2 + \varpi_C \hat{C}_t^2 + \varpi_L \hat{L}_t^2 + \varpi_\pi \hat{\pi}_t^2 + \mathbb{E}_f[\varpi_{\delta,f}(\partial_t \delta_{ft})^2] + \varpi_V \mathrm{Var}_f[\delta_{ft}] \\ -2\int_0^t \rho_s ds(\frac{C}{Y}\hat{C}_t - (1-\alpha)\hat{L}_t) - 2\hat{A}_t(\alpha \hat{K}_t + (1-\alpha)\hat{L}_t) \Bigg] \tag{S77}$$

with

$$\varpi_{KL} = \alpha(1-\alpha), \quad \varpi_{IK} = \phi\frac{K}{Y}, \quad \varpi_C = \sigma\frac{C}{Y}, \quad \varpi_L = \nu(1-\alpha), \tag{S78}$$

$$\varpi_\pi = \varepsilon\frac{1}{(\rho+\theta^p)\theta^p}, \quad \varpi_{\delta,f} = \frac{K}{Y}\frac{1}{\phi r^2}\frac{1}{(\rho+\theta_f^\delta)\theta_f^\delta}, \quad \varpi_V = \frac{K}{Y}\frac{1}{\phi r^2}, \tag{S79}$$

and $\mathbb{E}_f[x_f] = \sum_f \ell_f x_f$ and $\mathrm{Var}_f[x_f] \equiv \sum_f \ell_f[x_f - \mathbb{E}_f[x_f]]^2$ are expectation and variance operators for a group-specific variable x_f.

The optimal monetary policy problem is to minimize (S76) subject to the following log-linearized equilibrium conditions:

$$\begin{aligned}
C\hat{C}_t + I\hat{I}_t + Y\hat{G}_t &= Y\hat{A}_t + \alpha Y\hat{K}_t + (1-\alpha)Y\hat{L}_t \\
\partial_t \hat{K}_t &= (\iota_t - \xi) \\
\iota_t &= \frac{1}{\phi}\left[q_t - \frac{1}{r}(\delta_t - coc_t)\right] \\
\delta_t &= \mathbb{E}_f \delta_{ft} \\
\partial_t \delta_{ft} &= \theta(\delta_{ft}^* - \delta_{ft}) \\
\partial_t \delta_{ft}^* &= -(r+\theta_f^\delta)coc_t + r\delta_{ft}^* \\
\partial_t coc_{ft} &= -ri_t + rcoc_t \\
\partial_t q_t &= -(\omega_t - (i_t - \pi_t)) + rq_t \\
\partial_t \hat{C}_t &= -\frac{1}{\sigma}(i_t - \pi_t - \rho_t) \\
\partial_t \pi_t &= -(r+\theta^p)\theta^p\left[\hat{W}_t - \hat{P}_t - \alpha(\hat{K}_t - \hat{L}_t)\right] + r\pi_t
\end{aligned}$$

Since the objective function only involves second-order terms, the linear-quadratic problem provides a valid approximation to the original non-linear optimal monetary policy problem (Benigno and Woodford 2012).

Supplement A.8 Optimality Conditions for Monetary Policy

We derive the optimality conditions for the optimal monetary policy problem. For clarity, we assume there is one firm group, $F = 1$. The current value Hamiltonian is given by

$$\begin{aligned}
\mathcal{H} = \frac{1}{2}\Bigg[& \varpi_{KL}(\hat{L}_t - \hat{K}_t)^2 + \varpi_{IK}\iota_t^2 + \varpi_C \hat{C}_t^2 + \varpi_L \hat{L}_t^2 + \varpi_\pi \hat{\pi}_t^2 \\
& + \varpi_\delta (\theta_f^\delta)^2 (\delta_{ft}^* - \delta_{ft})^2 \\
& - 2\int_0^t \rho_s ds (\frac{C}{Y}\hat{C}_t - (1-\alpha)\hat{L}_t) \\
& - 2Y\hat{A}_t(\alpha\hat{K}_t + (1-\alpha)\hat{L}_t)\Bigg] \\
& + \lambda_{q,t}\left[\rho\hat{q}_t - \omega\left[-(\hat{K}_t - \hat{L}_t) + \left(\sigma\hat{C}_t + \nu\hat{L}_t\right)\right] + i_t - \pi_t\right] \\
& + \lambda_{\iota,t}\left[\iota_t - \frac{1}{\phi}\left[\hat{q}_t - \frac{1}{\rho}(\delta_t - coc_t)\right]\right] \\
& + \lambda_{C,t}\left[\frac{1}{\sigma}(i_t - \pi_t - \rho_t)\right] \\
& + \lambda_{\pi,t}\left[\rho\pi_t - \theta^p(\rho + \theta^p)\left[\sigma\hat{C}_t + \nu\hat{L}_t - \alpha(\hat{K}_t - \hat{L}_t)\right]\right] \\
& + \lambda_{RC,t}\left[\hat{A}_t + \alpha\hat{K}_t + (1-\alpha)\hat{L}_t - \left(\frac{C}{Y}\hat{C}_t + \frac{K}{Y}(\iota_t + \iota\hat{K}_t)\right)\right] \\
& + \lambda_{K,t}\left[\iota_t\right] \\
& + \lambda_{coc,t}\left[-ri_t + rcoc_t\right] \\
& + \lambda_{*,t}(\rho + \theta_f^\delta)\left[\delta_{ft}^* - coc_t\right] \\
& + \lambda_{\delta,t}\left[\theta_f^\delta(\delta_{ft}^* - \delta_{ft})\right].
\end{aligned}$$

Note that the initial conditions for $\lambda_{q,0}, \lambda_{\iota,0}, \lambda_{C,0}, \lambda_{\pi,0}, \lambda_{coc,t}$, and $\lambda_{*,t}$ are given.

The first-order optimality conditions are

$$L: \quad \varpi_{KL}(\hat{L}_t - \hat{K}_t) + \varpi_L \hat{L}_t + \int_0^t \rho_s ds(1-\alpha) - \hat{A}_t(1-\alpha) \tag{S80}$$

$$- \omega\lambda_{q,t}(1+\nu) - \lambda_{\pi,t}\theta^p(\rho+\theta^p)\nu + \lambda_{RC,t}(1-\alpha) = 0 \tag{S81}$$

$$K: \quad -\varpi_{KL}(\hat{L}_t - \hat{K}_t) - \hat{A}_t\alpha + \omega\lambda_{q,t} \tag{S82}$$

$$+ \theta^p(\rho+\theta^p)\alpha\lambda_{\pi,t} + \lambda_{RC,t}\left(\alpha - \frac{K}{Y}\iota\right) = \rho\lambda_{K,t} - \partial_t\lambda_{K,t} \tag{S83}$$

$$\iota: \quad \varpi_{IK}\iota_t + \lambda_{\iota,t} - \lambda_{RC,t}\frac{K}{Y} + \lambda_{K,t} = 0 \tag{S84}$$

$$C: \quad \varpi_C\hat{C}_t - \omega\sigma\lambda_{q,t} - \int_0^t \rho_s ds\frac{C}{Y} - \lambda_{\pi,t}\theta^p(\rho+\theta^p)\sigma - \lambda_{RC}\frac{C}{Y} = \rho\lambda_{C,t} - \partial_t\lambda_{C,t} \tag{S85}$$

$$\pi: \quad \varpi_\pi\hat{\pi}_t - \lambda_{q,t} - \lambda_C\frac{1}{\sigma} + \lambda_{\pi,t}\rho = \rho\lambda_{\pi,t} - \partial_t\lambda_{\pi,t} \tag{S86}$$

$$\lambda_q: \quad \lambda_{q,t}\rho - \lambda_\iota\frac{1}{\phi} = \rho\lambda_{q,t} - \partial_t\lambda_{q,t} \tag{S87}$$

$$i: \quad \lambda_{q,t} + \lambda_{C,t}\frac{1}{\sigma} - \rho\lambda_{coc,t} = 0 \tag{S88}$$

$$coc: \quad -\lambda_{\iota,t}\frac{1}{\phi\rho} + \rho\lambda_{coc,t} - \lambda_{*,t}(\rho+\theta_f^\delta) = \rho\lambda_{coc,t} - \partial_t\lambda_{coc,t} \tag{S89}$$

$$\delta_{ft}^*: \quad \varpi_\delta(\theta_f^\delta)^2(\delta_{ft}^* - \delta_{ft}) + \lambda_{*,t}(\rho+\theta_f^\delta) + \lambda_{\delta,t}\theta_f^\delta = \rho\lambda_{*,t} - \partial_t\lambda_{*,t} \tag{S90}$$

$$\delta_{ft}: \quad -\varpi_\delta(\theta_f^\delta)^2(\delta_{ft}^* - \delta_{ft}) + \lambda_{\iota,t}\frac{1}{\phi\rho} - \lambda_{\delta,t}\theta_f^\delta = \rho\lambda_{\delta,t} - \partial_t\lambda_{\delta,t}. \tag{S91}$$

Combining (S87) and (S88), we have

$$\varpi_\pi\hat{\pi}_t = \rho\lambda_{coc,t} - \partial_t\lambda_{\pi,t}. \tag{S92}$$

From this expression, if the economy is to have zero inflation in the long-run steady state (in which case $\partial_t\lambda_{\pi,t} = 0$), then it must be $\lambda_{coc,t} \to 0$ as $t \to \infty$.

Iterating (S91) forward, we have

$$\lambda_{\delta,t} = \int_t^\infty e^{-(\rho+\theta_{f,t}^\delta)(s-t)}\left[-\varpi_\delta(\theta_f^\delta)^2(\delta_{fs}^* - \delta_{fs}) + \lambda_{\iota,s}\frac{1}{\phi\rho}\right]ds. \tag{S93}$$

Iterating (S90) backward,

$$\lambda_{*,t} = -\int_0^t e^{\theta_f^\delta(s-t)}\left[\varpi_\delta(\theta_f^\delta)^2(\delta_{fs}^* - \delta_{fs}) + \lambda_{\delta,s}\theta_f^\delta\right]ds. \tag{S94}$$

Iterating (S89) backward,

$$\lambda_{coc,t} = \int_0^t \left[\lambda_{\iota,s} \frac{1}{\phi\rho} + \lambda_{*,t}(\rho + \theta_f^\delta) \right] ds. \tag{S95}$$

In the limit with a flexible discount rate, $\theta_f^\delta \to \infty$, we have

$$\lim_{\theta_f^\delta \to \infty} \lambda_{*,t}(\rho + \theta_f^\delta) + \lambda_{\iota,t} \frac{1}{\phi\rho} = 0, \tag{S96}$$

and therefore,

$$\lambda_{coc,t} = 0 \tag{S97}$$

for all t. This confirms existing results that long-run inflation must be zero (Woodford 2003). Outside of such a limit, it is generically not possible to have (S96). Consequently,

$$\lim_{t \to \infty} \lambda_{coc,t} = \int_0^\infty \left[\lambda_{\iota,s} \frac{1}{\phi\rho} + \lambda_{*,t}(\rho + \theta_f^\delta) \right] ds. \tag{S98}$$

This is the undiscounted sum of the past realizations of $\lambda_{\iota,s} \frac{1}{\phi\rho} + \lambda_{*,t}(\rho + \theta_f^\delta)$ in the history. Outside of the knife-edge case, we would not expect this term to be zero, which we numerically confirm.

Supplement B Hand-to-Mouth Households

We extend the baseline New Keynesian model by adding hand-to-mouth households who consume all their income every period. We largely follow Dupraz (2025) and adapt that model to continuous time.

Model Setup. A fraction $\chi \in [0, 1)$ of households are hand-to-mouth and the remaining households are Ricardian. Let y_t^r denote the flow income of the Ricardian households at period t. Let y_t^h denote the flow income of the hand-to-mouth households. The per-capita consumption function of the Ricardian household is, to a first-order approximation around the steady state,

$$\hat{C}_t^r = -(1/\sigma)\int_t^\infty e^{-r(s-t)}(r_s - \rho_s)ds + r\int_t^\infty e^{-r(s-t)}\hat{y}_s^r ds. \tag{S99}$$

The per-capita consumption of the hand-to-mouth households is simply

$$\hat{C}_t^h = \hat{y}_t^h. \tag{S100}$$

Aggregate consumption is then

$$\hat{C}_t = \chi\frac{y^h}{C}\hat{C}_t^h + (1-\chi)\frac{y^r}{C}\hat{C}_t^r \tag{S101}$$

$$= \chi\frac{y^h}{C}\hat{y}_t^h + (1-\chi)\frac{y^r}{C}\left[-(1/\sigma)\int_t^\infty e^{-r(s-t)}(r_s - \rho_s)ds + r\int_t^\infty e^{-r(s-t)}\hat{y}_s^r ds\right]. \tag{S102}$$

As in Werning (2015) and Dupraz (2025), we postulate the income of each household as a function of aggregate variables. In particular, we assume that both households earn a share $1-\alpha$ of aggregate income through labor income. The remaining share α of aggregate income goes to Ricardian households through capital income. These assumptions imply

$$y_t^r = (1-\alpha)(C_t + I_t) + \frac{1}{1-\chi}\left[\alpha(C_t + I_t) - I_t\right], \tag{S103}$$

$$y_t^h = (1-\alpha)(C_t + I_t). \tag{S104}$$

Log-linearizing,

$$y^r\hat{y}_t^r = \left[(1-\alpha) + \frac{1}{1-\chi}\alpha\right] C\hat{C}_t + (1-\alpha)\left[1 - \frac{1}{1-\chi}\right] I\hat{I}_t, \tag{S105}$$

$$y^h\hat{y}_t^h = (1-\alpha)C\hat{C}_t + (1-\alpha)I\hat{I}_t. \tag{S106}$$

Substituting (S105) and (S106) into (S102),

$$\begin{aligned}\hat{C}_t = r&\left[(1-\alpha) + \frac{1}{1-\chi}\alpha\right](1-\chi)\int_t^\infty e^{-r(s-t)}\hat{C}_s ds + (1-\alpha)\chi\hat{C}_t \\ &+ r(1-\alpha)\left[1 - \frac{1}{1-\chi}\right](1-\chi)\frac{I}{C}\int_t^\infty e^{-r(s-t)}\hat{I}_s ds + (1-\alpha)\chi\frac{I}{C}\hat{I}_t \\ &+ (1-\chi)\frac{y^r}{C}\left[-(1/\sigma)\int_t^\infty e^{-r(s-t)}(r_s - \rho_s)ds\right].\end{aligned} \tag{S107}$$

Using integration by parts,

$$r\int_t^\infty e^{-r(s-t)}\hat{C}_s ds = \hat{C}_t + \left[\int_t^\infty e^{-r(s-t)}\partial_t\hat{C}_s ds\right], \tag{S108}$$

$$r\int_t^\infty e^{-r(s-t)}\hat{I}_s ds = \hat{I}_t + \left[\int_t^\infty e^{-r(s-t)}\partial_t\hat{I}_s ds\right]. \tag{S109}$$

Substituting (S108) and (S109) into (S107),

$$\begin{aligned}\hat{C}_t = &\left[(1-\alpha) + \frac{1}{1-\chi}\alpha\right](1-\chi)\left[\hat{C}_t + \left[\int_t^\infty e^{-r(s-t)}\partial_t\hat{C}_s ds\right]\right] + (1-\alpha)\chi\hat{C}_t \\ &+ (1-\alpha)\left[1 - \frac{1}{1-\chi}\right](1-\chi)\frac{I}{C}\left[\hat{I}_t + \left[\int_t^\infty e^{-r(s-t)}\partial_t\hat{I}_s ds\right]\right] + (1-\alpha)\chi\frac{I}{C}\hat{I}_t, \\ &+ (1-\chi)\frac{y^r}{C}\left[-(1/\sigma)\int_t^\infty e^{-r(s-t)}(r_s - \rho_s)ds\right],\end{aligned} \tag{S110}$$

which simplifies to

$$\begin{aligned}0 = &[(1-\alpha)(1-\chi)+\alpha]\left[\int_t^\infty e^{-r(s-t)}\partial_t\hat{C}_s ds\right] - (1-\alpha)\chi\frac{I}{C}\left[\int_t^\infty e^{-r(s-t)}\partial_t\hat{I}_s ds\right] \\ &+ \left\{[(1-\alpha)(1-\chi)+\alpha] - (1-\alpha)\chi\frac{I}{C}\right\}\left[-(1/\sigma)\int_t^\infty e^{-r(s-t)}(r_s - \rho_s)ds\right].\end{aligned} \tag{S111}$$

Differentiating both sides by t, we have

$$\partial_t \hat{C}_t = (1-\varsigma)\,\frac{1}{\sigma}(r_t - \rho_t) + \varsigma\,\partial_t \hat{I}_t, \tag{S112}$$

where

$$\varsigma \equiv \frac{(1-\alpha)\chi\frac{I}{C}}{(1-\alpha)(1-\chi)+\alpha} \geq 0. \tag{S113}$$

Equation (S112) replaces (24) in the main text. Note that without hand-to-mouth households, $\chi = 0$, we have $\varsigma = 0$ and (S112) collapses to (24). If $\chi > 0$, then $\varsigma > 0$. In this case, an increase in investment growth also increases consumption growth, a mechanism highlighted in Auclert et al. (2020), Bilbiie et al. (2022), and Dupraz (2025).

The rest of the model is unchanged relative to the baseline model. In particular, we assume that the same New Keynesian Phillips curve (26) holds with heterogeneous agents, as in Auclert et al. (2021) and McKay and Wolf (2022). The only new parameter value is the fraction of hand-to-mouth agents, χ. We set this value to 0.49, following Auclert et al. (2024). The remaining parameters are unchanged.

Impulse Responses. The key mechanisms driven by sticky discount rates continue to play an important role in a model with hand-to-mouth households.

Figure S1 shows impulse responses to an increase in the inflation target, analogous to Figure 7 in the baseline model. Both investment and consumption increase thanks to the investment-consumption feedback highlighted in Auclert et al. (2020), Bilbiie et al. (2022) and Dupraz (2025). Greater consumption, in turn, further increases investment through an increase in aggregate demand. Compared to the baseline model, the GDP response is therefore stronger in the model with hand-to-mouth households.

Figure S2 shows impulse responses to an increase in the real interest rate, analogous to Figure 8 in the baseline model. The impulse responses are similar to those in the baseline model.

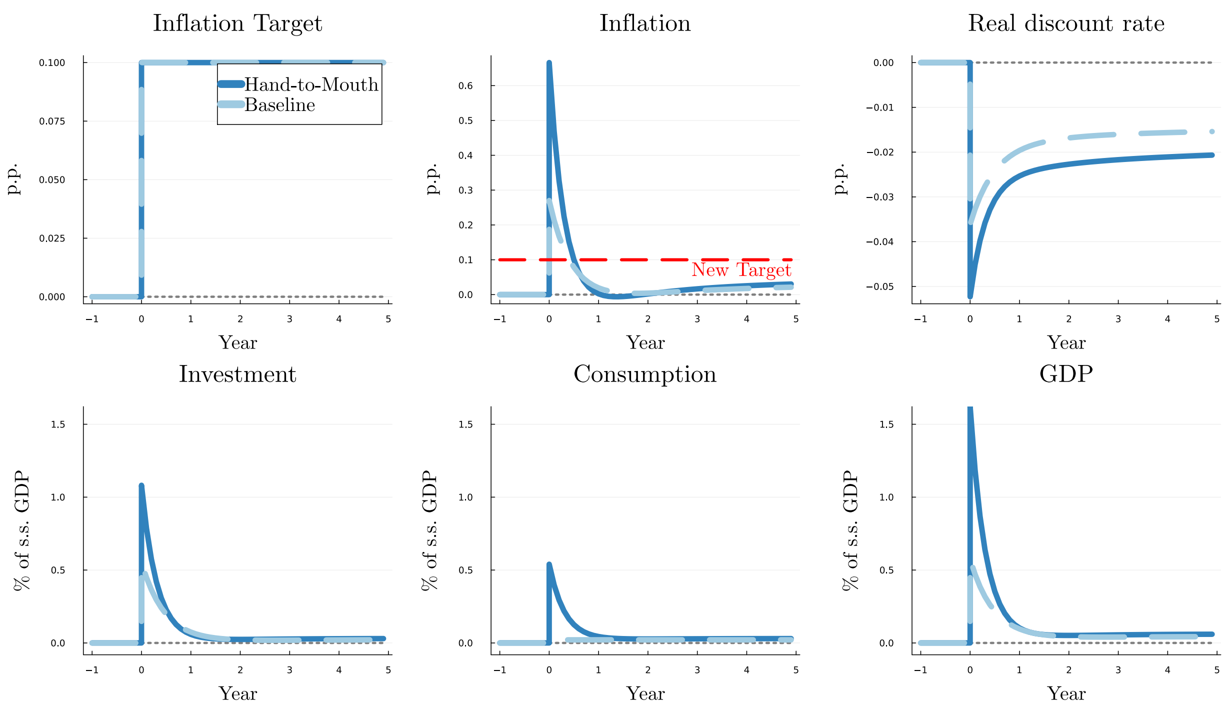

Figure S1: GE Responses to an Inflation Target Shock with Hand-to-Mouth Households

The figure plots aggregate impulse responses to a 0.1 percentage point increase in the inflation target with and without hand-to-mouth households. The inflation target, inflation, the real discount rate, and the real variables are annualized.

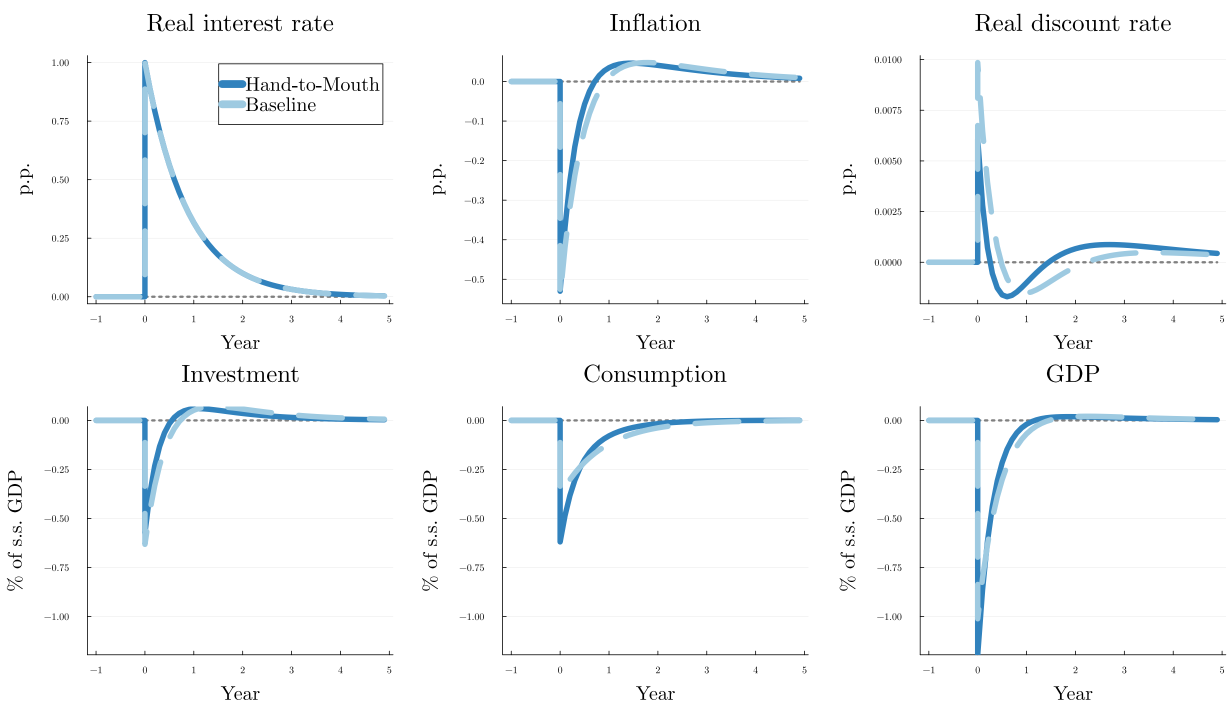

Figure S2: GE Responses to a Real Interest Rate Shock with Hand-to-Mouth Households

The figure plots aggregate impulse responses to a unit increase in the real interest rate that decays with a quarterly autocorrelation of 0.75 with and without hand-to-mouth households. The real interest rate, inflation, the real discount rate, and the real variables are annualized.

Supplement References

AUCLERT, A., M. ROGNLIE, M. SOUCHIER, AND L. STRAUB (2021): "Exchange Rates and Monetary Policy with Heterogeneous Agents: Sizing up the Real Income Channel," .

AUCLERT, A., M. ROGNLIE, AND L. STRAUB (2020): "Micro Jumps, Macro Humps: Monetary Policy and Business Cycles in an Estimated HANK Model," .

——— (2024): "The Intertemporal Keynesian Cross," *Journal of Political Economy*, 132, 4068–4121.

BENIGNO, P. AND M. WOODFORD (2012): "Linear-Quadratic Approximation of Optimal Policy Problems," *Journal of Economic Theory*, 147, 1–42.

BILBIIE, F. O., D. R. KÄNZIG, AND P. SURICO (2022): "Capital and Income Inequality: An Aggregate-Demand Complementarity," *Journal of Monetary Economics*, 126, 154–169.

DUPRAZ, S. (2025): "Putting the I Back in the IS Curve," .

MCKAY, A. AND C. WOLF (2022): "Optimal Policy Rules in HANK," .

WERNING, I. (2015): "Incomplete Markets and Aggregate Demand," .

WOODFORD, M. (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton, NJ: Princeton University Press.